CMS/ACM 107 Introduction to Linear Analysis with Applications

Andrew Stuart
astuart@caltech.edu, Annenberg 302.
Office Hours: by appointment.

TA: Matt Dunlop mdunlop@caltech.edu, Annenberg 334. Recitation: Monday 11-12, Annenberg 106. Office Hours: Wednesday 11-12 or by appointment, Annenberg 334.

Location and Time The course will meet in Steele 102 during the first week of classes, 9/27 and 9/29; it will resume in Firestone 384 on 10/4 and remain in that location thereafter. The lectures take place 10:30–12:00.

Prerequisites Applied linear algebra (Ma 1b, analytic track; ACM 104; or equivalent).

Rationale Linear algebra and linear analysis are basic building blocks required for the formulation and computational solution of many problems arising in science and engineering. The goals of this course are to: (i) learn the relevant concepts in these field, and how they fit together, and how they relate to other parts of mathematics; (ii) learn how to prove the basic results in these fields, not only because proofs are both intrinsically beautiful and provide interesting intellectual challenges, but also because they aid in understanding of the meaning and utility of relevant concepts; (iii) learn how to apply these mathematical tools to formulate problems arising in applications from data science, control and dynamical systems and differential equations; (iv) learn how to use the tools to solve problems, using computational algorithms, arising in these application domains.

Syllabus Covers the basic algebraic, geometric, and topological properties of normed linear spaces, inner-product spaces, and linear maps. Emphasis is placed both on rigorous mathematical development and on applications to control theory, data analysis and partial differential equations.

1. **Six Lectures** Finite dimensions: norms and inner-products on vectors and matrices; norm equivalence; eigenvalues and eigenvectors, matrix factorizations including normal diagonalization, Jordan form, Cholesky and the singular value decomposition; four subspace theorem; pseudo-inverse.

2. **Four Lectures** Banach and Hilbert spaces: linear operators, dual spaces, norms and inner-products, convexity, closest point, orthogonality, projection and least squares. ℓ^p spaces, density, Schauder bases and separability. Continuous and compact embeddings. Sequential compactness. Reisz representation. Lax-Milgram.

3. **Four Lectures** Functional Analysis: Banach algebras, contraction mapping principle, boundedness and continuity of linear maps, inverses of linear maps and Neumann series, adjoints. Spectral theorem for compact normal operators. Functional calculus. Positive operators. Semidefinite order. Topological dual of a normed space. Hahn-Banach theorem and consequences. Separation by hyperplanes and consequences.

4. **Four Lectures** Function Spaces: continuous and differentiable functions, extreme value and boundedness theorems, L^p spaces, Fourier transform, Sobolev spaces.

5. **One Lecture** Revisit, Revise, Recap, Redux.

Books The instructor will provide draft pdf lecture notes at the start of the course; these will not be perfect, but will be improved during the course and a revised set of notes made available at the end of term. The lectures may cover the material in a different order from the notes, or different proofs may be presented. Furthermore I will expect you to read and absorb the material in the notes, even if I do not cover it in the lectures. Any part of the final set of lecture notes, provided at the end of term, may be on the final exam.

The first chapter of the course is closely related to the first two chapters of the notes here:

`http : //www.seehuhn.de/pages/numlinalg`

and you may find that entire set of notes useful for broader context. Chapters 2 and 3 are based around linear functional analysis, and there are numerous books and online resources for this subject. Books I have personally benefitted from in this area include [6], [3], [4] and [5]; but you will probably find your own personal favourites. Chapter 4 concerns a subset of linear functional analysis related to functions defined on subsets of R^d. This will be touched on in the books just cited, but you will find more on the subject in books of differential equations. Parts of the comprehensive text by Evans [2] will be useful for context in this chapter, and the text [1] has a deep presentation of some of the function spaces that we will study, going far beyond our terse presentation, but giving useful context.

Prequisites

- **Ma1b, Analytical: Linear Algebra** This is a one term course on linear algebra over the real and complex numbers. Vector spaces are defined axiomatically and proofs are supplied for most results. Topics include: vector spaces and inner product spaces; subspaces, dimension, and bases; linear transformations; systems of linear equations; matrices; determinants; eigenvalues and eigenvectors; the characteristic polynomial; symmetric, hermitian, and unitary matrices and transformations.

- **ACM104 Applied Linear Algebra**. This is an intermediate linear algebra course aimed at a diverse group of students, including junior and senior majors in applied mathematics, sciences and engineering. The focus is on applications. Matrix factorizations play a central role. Topics covered include linear systems, vector spaces and bases, inner products, norms, minimization, the Cholesky factorization, least squares approximation, data fitting, interpolation, orthogonality, the QR factorization, ill-conditioned systems, discrete Fourier series and the fast Fourier transform, eigenvalues and eigenvectors, the spectral theorem, optimization principles for eigenvalues, singular value decomposition, condition number, principal component analysis, the Schur decomposition, methods for computing eigenvalues, non-negative matrices, graphs, networks, random walks, the Perron-Frobenius theorem, PageRank algorithm.

Course Assessment

- 10% **from weekly exercises sheets (Mathematics only).** Assigned on Friday of Week N. Due on Friday of Week N+1. (In the first week assigned in the first lecture.)

- 40% **from four assignments. (Mathematics and Matlab based).** (Dates Below).

- 10% **from five random quizes in class.** (5% **simply for participating.)**

- 40% **from one closed book final exam.** This will take place in the exam period December 7th–9th, details to be announced.

- Exercises, Assignments and Final Exam to be handed-in to box outside Annenberg 334 by **12 noon** on the due date.

- Lecture Notes, Exercises, Assignments and Final Exam will all be assigned through

  `http://www.mdunlop.org`

Assignments

- **1** Out 10/4/16. In 10/13/16.
- **2** Out 10/18/16. In 10/27/16.
- **3** Out 11/8/16. In 11/17/16.
- **4** Out 11/22/16. In 12/1/16.

References

[1] Robert A Adams and John JF Fournier. *Sobolev spaces*, volume 140. Academic press, 2003.

[2] LC Evans. Partial differential equations. graduate studies in mathematics, 19 (1998). *American Mathematical Society.*

[3] David H Griffel. *Applied functional analysis.* Courier Corporation, 2002.

[4] Vivian Hutson, J Pym, and M Cloud. *Applications of functional analysis and operator theory*, volume 200. Elsevier, 2005.

[5] David G Luenberger. *Optimization by vector space methods.* John Wiley & Sons, 1969.

[6] Eberhard Zeidler. *Applied functional analysis, Main principles and their application.* Contents of AMS, 1995.