ACM/CMS 107 Linear Analysis & Applications **Fall 2017**

Assignment 3: Linear ODEs and Control Theory **Due:** 1st December 2017

Introduction

Systems of ordinary differential equations (ODEs) can be used to describe many physical processes, from the dynamics of populations to the trajectories of missiles. The general form for such systems is as follows:

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t) = F(x(t), t), \quad x(0) = x_0,$$

where $x(t) \in \mathbb{R}^n$ for each $t \geq 0$. Note that a higher order system can always be written as a first order system in a higher dimensional space. The behavior of the solution $x(t)$ may be undesirable, and so it will often be of interest to control the system by, for example, forcing it. The above system is therefore generalized to

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t) = F(x(t), t, u(t)), \quad x(0) = x_0,$$

where $u : [0, \infty) \to \mathbb{R}$ is a control function. The function u may also depend on the state $x(t)$ at time t, in addition to t, to allow for feedback into the system. The question arises of how the control u can be chosen to enforce certain behavior of the solution x.

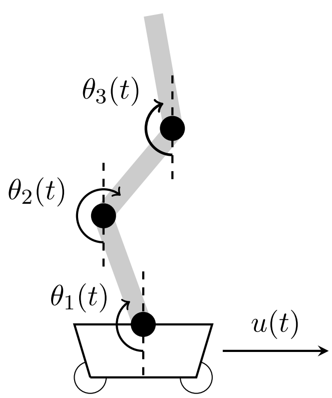

Figure 1: The triple pendulum on a cart.

As an example, consider a triple pendulum on a cart as illustrated in Figure 1. The state of the pendulum can be characterized by the pivot angles $(\theta_1, \theta_2, \theta_3)$. Under the laws of classical mechanics, the evolution of the angles $(\theta_1, \theta_2, \theta_3)$ in time is deterministic, though their behavior is chaotic and the system of differential equations that describes their evolution is nonlinear. It may be of interest to ask how the cart should be moved, according to the control $u(t)$, to ensure that each $\theta_i(t) \to \pi$ in a finite time so that the pendulum stands up straight. The following video illustrates the calculation and implementation of such a control:

https://www.youtube.com/watch?v=cyN-CRNrb3E

In Problem 1 we show how solutions to general linear systems of ODEs can be found via use of the matrix exponential. In Problem 2 we introduce the notion of controllability, and look at conditions for when this holds for linear systems. In Problem 3 we consider partially observed systems, and look at when we can infer a system's dynamics from these observations using a consequence of controllability. Finally in Problem 4 we study the systems from Problem 3 numerically, along with the Lorenz '63 model, which, like the triple pendulum, is also a nonlinear and chaotic system.

Problem 1. Linear ODEs (20 points)

(a) Let $A \in \mathbb{R}^{n\times n}$. Define the exponential of A by the Taylor series

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k.$$

Show that for each $t \geq 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{tA}\right) = Ae^{tA} = e^{tA}A,$$

assuming any results about the exponential from lectures and problem sets.

(b) Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$.

(i) Using part (a), write down the solution to the linear system

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t) = Ax(t), \quad x(0) = x_0.$$

If $x_0 \in \mathbb{C}^n$ is an eigenvector of A with eigenvalue $\lambda \in \mathbb{C}$, describe the behavior of $x(t)$ based on the values of real and imaginary parts of λ.

(ii) Let $u : [0, \infty) \to \mathbb{R}^m$ be a control function. Consider the forced linear system given by

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t) = Ax(t) + Bu(t), \quad x(0) = x_0.$$

The solution to this system is given by

$$x(t) = e^{tA}x_0 + \int_0^t e^{(t-s)A}Bu(s)\,\mathrm{d}s.$$

By use of a suitable integrating factor, derive this solution.

Problem 2. Controllability (25 points)

We define the space of *unrestricted controls* $\mathcal{U}$ as the set of functions $u : [0, \infty) \to \mathbb{R}^m$:

$$\mathcal{U} = \{u : [0, \infty) \to \mathbb{R}^m\}.$$

It is often of interest in control theory to work with a subset $\mathcal{U}_{ad} \subset \mathcal{U}$, referred to as the set of *admissible controls*, however in this assignment we work only with unrestricted controls.

Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. Given a control $u \in \mathcal{U}$, consider the *response* $x : [0, \infty) \to \mathbb{R}^n$ defined via the autonomous system

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \tag{1}$$

for some $x_0 \in \mathbb{R}^n$. If $u(\cdot)$ depends only upon the initial condition x_0, it is called an *open-loop control*. If it depends on the path $x(\cdot)$, it it called a *closed-loop control*. A closed-loop control allows feedback into the system, whereas an open-loop control does not.

We introduce the notion of *controllability* for this system:

Definition (Controllability)**.** *For each time $t > 0$, the* fixed time t controllable set *is defined by*

$$\mathcal{C}(t) = \{x_0 \in \mathbb{R}^n : \textit{there exists } u \in \mathcal{U} \textit{ with } x(t) = 0\}.$$

The controllable set *is defined by*

$$\mathcal{C} = \bigcup_{t>0} \mathcal{C}(t).$$

If $\mathcal{C} = \mathbb{R}^n$, we will say that the system is controllable. *We will alternatively say that (A, B) is controllable in this case.*

The system is hence controllable if it can be controlled to hit zero in a finite time, from any starting point. A useful characterization of controllability is given in terms of the *controllability matrix* $G(A, B) \in \mathbb{R}^{n \times mn}$ of (1). This is defined by

$$G(A, B) = \big(B, AB, A^2B, \ldots, A^{n-1}B\big).$$

Theorem. *(A, B) is controllable if and only if* $\operatorname{rank}\big(G(A, B)\big) = n$.

Proof of this theorem may be found in, for example, [1].

(a) The Cayley-Hamilton theorem states that every square matrix satisfies its own characteristic polynomial. Using this, explain why

$$\operatorname{rank}\big(G(A, B), A^kB\big) = \operatorname{rank}\big(G(A, B)\big)$$

for any $k \geq n$.

(b) Let $n = m = 2$.

(i) Let $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Under what conditions on A is the system controllable?

(ii) Let $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Under what conditions on A is the system controllable?

(c) Let $n = 3$, $m = 1$, and define $A \in \mathbb{R}^{n \times n}$ by

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}.$$

Consider the three cases

$$B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \text{ and } \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

For which of these choices of B is the system controllable? In the case(s) where the system is not controllable, find the controllable set $\mathcal{C}$.

Problem 3. Observer (25 points)

Suppose that we wish to recover a signal $x(t)$ that is known to satisfy the system

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t) = Ax(t), \quad x(0) = x_0,$$

but we do not know the initial condition x_0. We do not observe $x(t)$ directly, but instead have observations given by

$$y(t) = Hx(t)$$

for some $H \in \mathbb{R}^{m\times n}$. We wish to use these observations to determine $x(t)$ for sufficiently large t. To this end, we consider the system given by

$$\frac{\mathrm{d}z}{\mathrm{d}t}(t) = Az(t) + B\big(y(t) - Hz(t)\big), \quad z(0) = z_0 \tag{2}$$

for some $B \in \mathbb{R}^{n\times m}$. This is of the form (1) with the closed loop control $u = y - Hz$. The reasoning for this is that when z is close to x, so that y is close to Hz, the dynamics of z should be close to those of x. In particular, if $z(t_*) = x(t_*)$ for some $t_* \geq 0$, then $z(t) = x(t)$ for all $t \geq t^*$. In general we cannot guarantee that the positions of z and x will ever coincide, but we can instead aim for $z(t)$ and $x(t)$ to become arbitrarily close for large times t.

(a) Define the error $e(t) = z(t) - x(t)$. Show that $e(t)$ satisfies the equation

$$\frac{\mathrm{d}e}{\mathrm{d}t}(t) = (A - BH)e(t), \quad e(0) = z_0 - x_0$$

and hence give criterion that ensure $e(t) \to 0$ as $t \to \infty$ for any choice of z_0.

(b) If (A, B) is controllable, it is known that there exists an observation matrix $H \in \mathbb{R}^{m\times n}$ such that $e(t) \to 0$.

(i) Consider the cases from Problem 2(b)-(c) which were controllable. By hand, find observation matrices H such that the $e(t) \to 0$.

(ii) In the cases from Problem 2(b)-(c) where (A, B) were not controllable, can you find $H \in \mathbb{R}^{m\times n}$ such that $e(t) \to 0$ for any z_0? Investigate either numerically or by hand, choosing at least one example from each of 2(b) and 2(c) where controllability does not hold, and looking at the spectrum of $A - BH$ for different choices of H.

Remark. *Above we are given the matrices A, B, and use controllability to see that we can choose an observation matrix H that causes $e(t) \to 0$. There is also a dual notion to controllability, called observability. If we are instead given the matrices A, H, and the pair (A, H) is observable, then it can be shown that we can choose a matrix B such that $e(t) \to 0$.*

Problem 4. Numerics (30 points)

(a) Implement the the systems (1), (2) in MATLAB, for arbitrary $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. You can use the forms of the solutions given in Problem 1, solve using one of MATLAB's built-in ODE solvers such as `ode45`, or implement a Runge-Kutta method directly.

(i) Choose an example from Problem 2(b) where you know that $e(t) \to 0$ for any z_0. Fix x_0, and verify that this is the case by plotting (on the same axes) the trajectories of $e(\cdot)$ for variety of choices of z_0.

(ii) Let $n = m = 3$. Define $A \in \mathbb{R}^{3\times 3}$ by

$$A = \begin{pmatrix} 0.1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

and let $B = I$. Observe that $\operatorname{rank}\big(G(A,B)\big) = 3$, and so there exists an observation matrix $H \in \mathbb{R}^{3\times 3}$ that drives the error to zero.

Fix $x_0 = (1, 1, 0)$ and $z_0 = (2, 2, 2)$. Consider the three observation matrices H given by

$$H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

For each H, plot the error $\varepsilon(t) = \|e(t)\| = \|x(t) - z(t)\|$ for $t \in [0, 50]$. By considering the trajectories of $x(\cdot)$ and $z(\cdot)$, and the structures of A, B, H, explain the behavior of these errors.

(b) We now consider a nonlinear example. The Lorenz '63 model is defined as follows:

$$\begin{aligned} \frac{\mathrm{d}x_1}{\mathrm{d}t}(t) &= \sigma\big(x_2(t) - x_1(t)\big) \\ \frac{\mathrm{d}x_2}{\mathrm{d}t}(t) &= x_1(t)\big(\rho - x_3(t)\big) - x_2(t) \\ \frac{\mathrm{d}x_3}{\mathrm{d}t}(t) &= x_1(t)x_2(t) - \beta x_3(t) \end{aligned}$$

where σ, ρ, β are scalar parameters and $x_1(t), x_2(t), x_3(t) \in \mathbb{R}$ for each t. In what follows we will fix $\sigma = 10$, $\beta = 8/3$ and $\rho = 28$; it is known that with these choices the system is chaotic.

We can write the above system more compactly as

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t) = F(x(t)), \quad x(0) = \big(x_1(0), x_2(0), x_3(0)\big). \tag{3}$$

where now $F : \mathbb{R}^3 \to \mathbb{R}^3$ is nonlinear. Suppose that we have observations

$$y(t) = Hx(t)$$

for some $H \in \mathbb{R}^{3\times 3}$, and analogously to the linear case (2), consider the controlled system

$$\frac{\mathrm{d}z}{\mathrm{d}t}(t) = F(z(t)) + B\big(y(t) - Hz(t)\big), \quad z(0) = \big(z_1(0), z_2(0), z_3(0)\big). \tag{4}$$

for some $B \in \mathbb{R}^{3\times 3}$.

(i) Implement both systems (3) and (4) in MATLAB, for arbitrary observation and control matrices H, B. Plot the solution to (3) for a number of initial conditions and $t \in [0, 50]$. Observe the sensitivity of the solution to the choice of initial condition: two distinct but arbitrarily close initial conditions can lead to very different trajectories.

(ii) We consider the case where we only observe one component of the solution, so that $H \in \mathbb{R}^{3\times 3}$ is given by

$$H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Additionally we assume $B = \alpha I$ for some $\alpha > 0$, where I is the identity matrix, and consider $t \in [0, 50]$.

Fix $x(0)$ and $z(0)$, with $\|x(0) - z(0)\| \geq 10$. Define the error $\varepsilon(t) = \|x(t) - z(t)\|$. For each choice of H above, can you find $\alpha > 0$ such that that $\varepsilon(t) \approx 0$ for large t? If this is the case, how small can you take α for this to hold? Produce a plot of the error $\varepsilon(t)$ for each choice of H, with $\varepsilon(t)$ becoming small if possible.

References

[1] L.C. Evans. *An Introduction to Mathematical Optimal Control Theory.* 2005. https://math.berkeley.edu/~evans/control.course.pdf.