CMS 107

CALIFORNIA INSTITUTE OF TECHNOLOGY

FINAL EXAMINATION: PART I

INTRODUCTION TO LINEAR ANALYSIS WITH APPLICATIONS

Out: Friday November 17th, 9am

Due: Tuesday November 21st, 5pm

Candidates should attempt **all** questions.

You may use ONLY your own notes. You may **not** use the online/printed lecture notes, books, or the internet.

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1. a) For every $s \in \mathbb{R}$ define:

$$\mathcal{X}^s = \left\{ v \in \mathbb{R}^\infty : \sum_{j=1}^{\infty} j^{2s} |v_j|^2 < \infty \right\}.$$

Show that this is a Hilbert space when equipped with the inner-product

$$\langle v, w \rangle_{\mathcal{X}^s} = \sum_{j=1}^{\infty} j^{2s} v_j w_j$$

and norm

$$\|v\|_{\mathcal{X}^s}^2 = \sum_{j=1}^{\infty} j^{2s} |v_j|^2.$$

[5]

b) For which s and r is $\mathcal{X}^s$ continuously embedded into ℓ^r? **[10]**

c) Prove that $\mathcal{X}^s$ is compactly embedded into ℓ^2 for every $s > 0$. Find a counter example to show that compact embedding does not hold for any $s < 0$. **[10]**

END