# CMS/ACM 107 Introduction to Linear Analysis with Applications

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Recitation: Monday 9-10, Annenberg 107.
Office Hours: Wednesday 9-10 or by appointment, Annenberg 107.

**Location and Time** Lectures take place in Firestone 384, Tuesday and Thursday, 10:30–12:00.

**Prerequisites** Applied linear algebra (Ma 1b, analytic track; ACM 104; or equivalent).

**Rationale** Linear algebra and linear analysis are basic building blocks required for the formulation and computational solution of many problems arising in science and engineering. The goals of this course are to: (i) learn the relevant concepts in these field, and how they fit together, and how they relate to other parts of mathematics; (ii) learn how to prove the basic results in these fields, not only because proofs are both intrinsically beautiful and provide interesting intellectual challenges, but also because they aid in understanding of the meaning and utility of relevant concepts; (iii) learn how to apply these mathematical tools to formulate problems arising in applications from data science, control and dynamical systems and differential equations; (iv) learn how to use the tools to solve problems, using computational algorithms, arising in these application domains.

**Syllabus** Covers the basic algebraic, geometric, and topological properties of normed linear spaces, inner-product spaces, and linear maps. Emphasis is placed both on rigorous mathematical development and on applications to control theory, data analysis and partial differential equations. The following is an approximate summary of the distribution of topics, and order of presentation.

1. **Three lectures** Topological, metric, normed vector spaces and inner-product spaces; Banach space and Hilbert spaces. Examples from finite dimensions, spaces of sequences, spaces of real-valued functions over a subset of Euclidean space. Fourier transform.

2. **Four lectures** Linear operators. Operator and matrix norms. Dual spaces. The adjoint. Density, Schauder bases and separability. Continuous embeddings. Compact operators and compact embeddings. Sobolev embeddings.

3. **Four lectures** Convexity, closest point, orthogonality, projection and least squares. Matrix and operator factorizations: Spectral theorem for compact normal operators, SVD, Jordan Normal Form. Functional calculus. $e^{At}$ and applications in control. Cayley-Hamilton Theorem. Sobolev-like weighted sequence spaces. $L_2$ based operator norms.

4. **Three lectures** Solving linear equations. Banach algebras, boundedness and continuity of linear maps, inverses of linear maps and Neumann series. Riesz representation theorem. Lax-Milgram lemma and applications in PDEs. Least squares. Contraction mapping principle.

5. **Three lectures** Ordinary differential equations. Finite dimensional setting. Hilbert space setting. Steepest descent and applications in data science.

**Books** The course will be linked to the set of lecture notes here:

`https://caltech.box.com/s/n1clhsvwsq648p97gq66t7d3nuib4hji`

However the lectures will present the material in a different order than that developed in these notes. Furthermore, the lectures will not cover all of the material in the notes, there will be additional material covered not in the notes, and different proofs may be presented. Exercises are provided in the lecture notes, along with worked solutions to a subset of these; I will point to relevant exercises for self study each week.

There are numerous books and online resources for this subject. Books I have personally benefited from in this area include [6], [3], [4] and [5]; but you will probably find your own personal favourites. Part of the course concerns a subset of linear functional analysis related to functions defined on subsets of $\mathbb{R}^d$. This will be touched on in the books just cited, but you will find more on the subject in books of differential equations. Parts of the comprehensive text by Evans [2] will be useful for context in this chapter, and the text [1] has a deep presentation of some of the function spaces that we will study, going far beyond our terse presentation, but giving useful context.

**Prerequisites**

- **Ma1b, Analytical: Linear Algebra** This is a one term course on linear algebra over the real and complex numbers. Vector spaces are defined axiomatically and proofs are supplied for most results. Topics include: vector spaces and inner product spaces; subspaces, dimension, and bases; linear transformations; systems of linear equations; matrices; determinants; eigenvalues and eigenvectors; the characteristic polynomial; symmetric, Hermitian, and unitary matrices and transformations.

- **ACM104 Applied Linear Algebra**. This is an intermediate linear algebra course aimed at a diverse group of students, including junior and senior majors in applied mathematics, sciences and engineering. The focus is on applications. Matrix factorizations play a central role. Topics covered include linear systems, vector spaces and bases, inner products, norms, minimization, the Cholesky factorization, least squares approximation, data fitting, interpolation, orthogonality, the QR factorization, ill-conditioned systems, discrete Fourier series and the fast Fourier transform, eigenvalues and eigenvectors, the spectral theorem, optimization principles for eigenvalues, singular value decomposition, condition number, principal component analysis, the Schur decomposition, methods for computing eigenvalues, non-negative matrices, graphs, networks, random walks, the Perron-Frobenius theorem, PageRank algorithm.

**Course Assessment**

- 45% **from three assignments. (Mathematics and MATLAB based).** (Dates below).

- 15% **from three randomly timed quizzes in class.** (5% **simply for participating.)**

- 40% **from midterm and final exams.** Midterm will be made available 9:00am November 17th and due 5:00pm November 21st. Final will be made available 9:00am December 1st and due 5:00pm December 5th. Both will be take-home exams, with several days to complete, but designed to take one hour. You may use your own lecture notes, but must not access any other material. Further details to be announced.

- Assignments to be handed-in to box outside Annenberg 337 by **12 noon** on the due date.

- Lecture notes and assignments will all be assigned through

  http://www.mdunlop.org

**Assignments**

- **1** Out 10/03/17. In 10/12/17.
- **2** Out 10/24/17. In 11/02/17.
- **3** Out 11/14/17. In 11/28/17.

# References

[1] Robert A Adams and John JF Fournier. *Sobolev spaces*, volume 140. Academic press, 2003.

[2] LC Evans. Partial differential equations. graduate studies in mathematics, 19 (1998). *American Mathematical Society.*

[3] David H Griffel. *Applied functional analysis.* Courier Corporation, 2002.

[4] Vivian Hutson, J Pym, and M Cloud. *Applications of functional analysis and operator theory*, volume 200. Elsevier, 2005.

[5] David G Luenberger. *Optimization by vector space methods.* John Wiley & Sons, 1969.

[6] Eberhard Zeidler. *Applied functional analysis, Main principles and their application.* Contents of AMS, 1995.