CMS 107

CALIFORNIA INSTITUTE OF TECHNOLOGY

FINAL EXAMINATION: PART II

INTRODUCTION TO LINEAR ANALYSIS WITH APPLICATIONS

Out: Friday December 1st, 9am

Due: Tuesday December 5th, 5pm

Candidates should attempt **all** questions.

You may use ONLY your own notes. You may **not** use the online/printed lecture notes, books, or the internet.

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1. a) Define the terms *symmetric*, *positive-definite* and *orthogonal* for real $n \times n$ matrices. **[5]**

b) State the factorization theorem for positive-definite symmetric real $n \times n$ matrices given in the lectures. **[5]**

c) Let $A \in \mathbb{R}^{n \times n}$ be a positive-definite symmetric matrix. Show that there is a unique positive-definite symmetric matrix $\sqrt{A}$ satisfying $\sqrt{A}\sqrt{A} = A$. **[5]**

d) Let $A \in \mathbb{R}^{n \times n}$ have positive determinant. Show that it is possible to factorize $A = QU$ where $U = \sqrt{(A^T A)}$ is positive-definite symmetric and Q is an orthogonal matrix with positive determinant. (This is known as the *polar decomposition.*) **[10]**

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2. a) State explicit formulae for the norms $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$ on real $n \times n$ matrices. Using these formulae calculate these norms for the matrix

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

[5]

b) Define a matrix norm in abstract. Given a vector norm, define the induced norm on matrices. **[5]**

c) Show that $\|AB\| \leq \|A\|\|B\|$ for any pair of $n \times n$ matrices A, B and any induced norm $\|\cdot\|$. **[5]**

d) Derive the formula for $\|\cdot\|_2$ from (a). **[10]**

END